Review for Intro. to Analysis Final Exam (Friday, May 11, 2007)

*An item marked with a * has a good chance of being on the final; with a -, not so good.*

Version of Friday, May 10, 9:30 AM

Triangle Inequality *10*

For all $a, b \in \mathbb{R}$, $\boxed{|a+b| \leq |a| + |b|}$. *(for Michelle)*

Theorem 1.7 (ε Has x's Back Against a Wall) *12*

For $x \in \mathbb{R}$, if $|x| < \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

Definition of Boundedness *21*

Let S be a nonempty set of real numbers.

1. S is bounded above iff there exists M such that for all $x \in S$, $x \leq M$. (M is an upper bound of S.)
2. S is bounded iff there exists M such that for all $x \in S$, $|x| \leq M$. (M is a bound of S.)

Definition of Supremum and Infimum *21*

Let S be a nonempty set of real numbers.

1. Suppose S is bounded above. A number β is the supremum of S if β is an upper bound of S and any number less than β is not an upper bound of S. ($\beta = \sup S$) *
2. Suppose S is bounded below. A number α is the infimum of S if α is a lower bound of S and any number greater than α is not a lower bound of S. ($\alpha = \inf S$)

Completeness Axiom *23* *

Each nonempty set of real numbers that is bounded above has a supremum.

Definition of Increasing/Decreasing and Monotone Functions *40* -

Let I be an interval, let $f : I \to \mathbb{R}$, and let J be a subinterval of I.

1. The function f is increasing on J if for all $x, y \in J$ that satisfy $x < y$, $f(x) \leq f(y)$.
2. The function f is monotone on J if it is either increasing or decreasing on J.

Definition of Bounded Functions *40*

Let I be an interval, let $f : I \to \mathbb{R}$, and let J be a subinterval of I.

1. The function f is bounded above on J if there exists a number M such that for all $x \in J$, $f(x) \leq M$.

2. The function f is bounded on J if there exists a number M such that for all $x \in J$, $|f(x)| \leq M$.

Definition of Extrema *41* -

Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$.

1. The function f has a maximum value at c if for all $x \in I$, $f(x) \leq f(c)$.
2. The function f has a relative maximum value at c if there exists $\delta > 0$ such that for all $x \in I$ that satisfy $|x - c| < \delta$, then $f(x) \leq f(c)$.

Definition of Sequence *50*

A sequence is a function whose domain is $\mathbb{N}$.

Definition of Bounded, Increasing/Decreasing, and Monotone Sequences *51* *

Let $\{x_n\}$ be a sequence of real numbers.

1. The sequence is bounded above if there exists M such that for all n, $x_n \leq M$.
2. The sequence is bounded if there exists M such that for all n, $|x_n| \leq M$.
3. The sequence is increasing if for all n, $x_n \leq x_{n+1}$.
4. The sequence is monotone of it is increasing or decreasing.

Definition of Convergence of a Sequence *52* *

A sequence $\{x_n\}$ converges to L if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \varepsilon$. The sequence diverges if it does not converge.

Theorem 2.4 (Uniqueness of Sequence Limits) *53*

The limit of a convergent sequence is unique.

Proof. Suppose $\{x_n\}$ converges to two limits, A and B. Choose $\varepsilon > 0$ arbitrary. By definition, there exists $N_A \in \mathbb{N}$ such that for all $n \geq N_A$, $|x_n - A| < \frac{\varepsilon}{2}$. Also, there exists $N_B \in \mathbb{N}$ such that for all $n \geq N_B$, $|x_n - B| < \frac{\varepsilon}{2}$. Then for $N = \max\{N_A, N_B\}$, by the Triangle Inequality,

$$|A - B| = |A - x_N + x_N - B| \leq |A - x_N| + |x_N - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $|A - B| < \varepsilon$ for all $\varepsilon > 0$, $|A - B| = 0 \Rightarrow A = B$. Thus, a sequence's limit is unique.

Theorem 2.5 (Convergent $\Rightarrow$ Bounded) *54*

Every convergent sequence is bounded.

Definition of Converging to ∞ *55*

A sequence $\{x_n\}$ of real numbers converges to ∞ if for all $M > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

Squeeze Theorem for Sequences *57* *

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and suppose that $\{x_n\}$ is a sequence such that for all n, $a_n \leq x_n \leq b_n$. If $\{a_n\}$ and $\{b_n\}$ both converge to L, then $\{x_n\}$ converges to L.

Proof. Choose $\varepsilon > 0$ arbitrary. By definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$. This implies that for all $n \geq N$, $b_n < L + \varepsilon$ and $L - \varepsilon < a_n$. Then

$$L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon \Rightarrow |x_n - L| < \varepsilon$$

Thus, $\{x_n\}$ converges to L.

Bounded/Monotone Convergence Theorem *61* *

A monotone sequence converges iff it is bounded.

Proof. ($\Rightarrow$). Recall that every convergent sequence is bounded. ($\Leftarrow$). Let $\{x_n\}$ be an increasing, bounded sequence. By the Completeness Axiom, there exists $\beta = \sup\{x_n \mid n \in \mathbb{N}\}$. Choose $\varepsilon > 0$ arbitrary. Since $\beta - \varepsilon$ isn't an upper bound, there exists $N \in \mathbb{N}$ such that $x_N > \beta - \varepsilon$. Since $\{x_n\}$ is increasing, for all $n \geq N$, $\beta - \varepsilon < x_N \leq x_n \leq \beta \leq \beta + \varepsilon$. Thus, $|x_n - \beta| < \varepsilon$. Therefore, $\{x_n\}$ converges to β.

Definition of Cauchy Sequence *63* *

A sequence $\{x_n\}$ is a Cauchy sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_m - x_n| < \varepsilon$.

Theorem 2.13 (Cauchy $\Leftrightarrow$ Convergent) *64* *

A sequence of real numbers converges iff it is a Cauchy sequence.

Definition of Subsequence *69*

Let $\{x_n\}$ be a sequence and let $\{p_n\}$ be a strictly increasing sequence of positive integers. The sequence $\{x_{p_n}\}$ is a subsequence of $\{x_n\}$.

Theorem 2.17 (Subsequences and Convergence) *69*

Let $\{x_n\}$ be a sequence of real numbers.

1. If $\{x_n\}$ converges to L, then every subsequence of $\{x_n\}$ converges to L.
2. If $\{x_n\}$ has two subsequences that converge to different limits, then $\{x_n\}$ does not converge.

Theorem 2.18 (Monotone Subsequence) *70*

Every sequence of real numbers has a monotone subsequence.

Bolzano-Weierstrass Theorem *70*

Every bounded sequence has a convergent subsequence.

Definition of Limit of a Function *82* *

Let I be an open interval with $c \in I$ and suppose that f is a function defined on I except possibly at c. The function f has a limit L at c if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in I$ that satisfy $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$.

Theorem 3.2 (Function/Sequence Convergence) *85* *
Let I be an open interval that contains the point c and suppose that f is a function that is defined on I except possibly at the point c. The function f has a limit L at c iff for each sequence $\{x_n\}$ in $I \setminus \{c\}$ that converges to c, the sequence $\{f(x_n)\}$ converges to L.

Squeeze Theorem for Functions *87*
Let I be an open interval that contains the point c and suppose that f, g, and h are functions that are defined on I except possibly at the point c. Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in $I \setminus \{c\}$. If $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$, then f has a limit at c and $\lim_{x\to c} f(x) = L$.

Definition (and Theorems) of Continuity *93, 94, 95*
Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. The function f is continuous at c if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in I$ that satisfy $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$. The function f is continuous on I if f is continuous at each point of I. [Sums, differences, products, quotients (where defined) and compositions (where defined) of continuous functions are continuous. Also, polynomial functions, rational functions, and the six trigonometric functions (where defined) are continuous.]

Intermediate Value Theorem *100* *
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. If K is a number between $f(a)$ and $f(b)$, then there is a point $c \in (a, b)$ such that $f(c) = K$.

Extreme Value Theorem *101* *
If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then there exist points c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

Definition of Intermediate Value Property *104*
A function f defined on an interval I has the intermediate value property on I if it satisfies the following condition: if a and b are distinct points in I and K is a number between $f(a)$ and $f(b)$, then there exists a point c between a and b such that $f(c) = K$.

Definition of Differentiable *131* *
Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. The function f is differentiable at c provided that the limit

$$\lim_{v\to c} \frac{f(v) - f(c)}{v - c}$$

exists. Then f has a derivative at c, denoted $f'(c)$. Also, f is differentiable on an interval J if f is differentiable at each point of J.

Theorem 4.2 (Continuous Seq. $\Leftrightarrow$ Differentiable Func.) *131* *
Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. The function f is differentiable at c with derivative $f'(c) = L$ iff for each sequence $\{x_n\}$ in $I \setminus \{c\}$ that converges to c, the sequence $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}$ converges to L.

Theorem 4.3 (Differentiable Implies Continuous) *132*
Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. If f is differentiable at c, then f is continuous at c. Consequently, if f is differentiable on an interval J, then f is continuous on J.

Product Rule *133* *

If f and g are differentiable functions defined on an interval I, then fg is differentiable on I and $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$.

Quotient Rule *134* *

If f and g are differentiable functions defined on an interval I and g is nonzero on I, then f/g is differentiable on I and

$$(f/g)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

for all $x \in I$.

Chain Rule *134* *

Let I be an interval, let $g : I \to \mathbb{R}$, and let f be a function defined on an interval J that contains $g(I)$. If g is differentiable on I and f is differentiable on J, then the function $f \circ g$ is differentiable on I and $(f \circ g)'(x) = f'(g(x))g'(x)$ for all $x \in I$.

Proof. Fix $c \in I$ and define a function $F : J \to \mathbb{R}$ by

$$F(x) = \begin{cases} \dfrac{f(x) - f(g(c))}{x - g(c)} & \text{if } x \neq g(c) \\ f'(g(c)) & \text{if } x = g(c) \end{cases}$$

Since f is differentiable at $g(c)$, the function F is continuous at $g(c)$. Since the equality

$$F(x)(x - g(c)) = f(x) - f(g(c))$$

is valid for all $x \in J$, it follows that

$$F(g(x))(g(x) - g(c)) = f(g(x)) - f(g(c))$$

for all $x \in I$. Since $F \circ g$ is continuous at c and g is differentiable at c, we have

$$\begin{aligned} (f \circ g)'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\ &= \lim_{x \to c} F(g(x)) \frac{g(x) - g(c)}{x - c} \\ &= F(g(c))g'(c) \end{aligned}$$

Since $F(g(c)) = f'(g(c))$, the proof is complete.

Theorem 4.10 (Extreme Value and Differentiability) *139*

Let I be an interval, let $f : I \to \mathbb{R}$, and suppose that $c \in I$ is not an endpoint of I. If f has a relative extreme value at c, then either f is not differentiable at c or $f'(c) = 0$.

Rolle's Theorem *140*

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem *141* *

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

L'Hôpital's Rule *141* *

Let f and g be continuous on $[a, b]$ and differentiable on (a, b) except possibly at the point $c \in (a, b)$, and suppose that $g' \neq 0$ on $(a, b) \setminus \{c\}$. If $\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x)$ and $\lim_{x \to c} f'(x)/g'(x)$ exists, then $\lim_{x \to c} f(x)/g(x)$ exists and

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Definition of Partition, Norm, and Refinement *166*

A partition P of an interval $[a, b]$ is a finite set of points $\{x_i \mid 0 \leq i \leq n\}$ such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. The norm of a partition P, denoted $\|P\|$, is the largest of the numbers $x_i - x_{i-1}$; that is, $\|P\| = \max\{x_i - x_{i-1} \mid 1 \leq i \leq n\}$. If P_1 and P_2 are partitions of $[a, b]$ and $P_1 \subseteq P_2$, then P_2 is a refinement of P_1.

Definition of Tagged Partition *166* *

A tagged partition ${}^tP$ of an interval $[a, b]$ consists of a partition $P = \{x_i \mid 0 \leq i \leq n\}$ of $[a, b]$ along with a set $\{t_i \mid 1 \leq i \leq n\}$ of points, called tags, that satisfy $x_{i-1} \leq t_i \leq x_i$ for $1 \leq i \leq n$; that is, ${}^tP = \{(t_i, [x_{i-1}, x_i]) \mid 1 \leq i \leq n\}$.

Definition of Riemann Sum *166*

Let $f : [a, b] \to \mathbb{R}$ and let ${}^tP = \{(t_i, [x_{i-1}, x_i]) \mid 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$. The Riemann sum $S(f, {}^tP)$ of f associated with ${}^tP$ is defined by

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Definition of Riemann Integrable *167* *

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if there exists a number L with the following property: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all tagged partitions ${}^tP$ of $[a, b]$ that satisfy $\|{}^tP\| < \delta$, $|S(f, {}^tP) - L| < \varepsilon$.

Definition of Oscillation *168* *

Let f be a bounded function defined on an interval $[a, b]$. The oscillation of f on $[a, b]$ is defined by $\omega(f, [a, b]) = \sup\{f(x) \mid x \in [a, b]\} - \inf\{f(x) \mid x \in [a, b]\}$.

Theorem 5.8 (Cauchy Criterion for Riemann Integrability) *172*

A bounded function f is Riemann integrable on $[a, b]$ iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all tagged partitions ${}^tP_1$ and ${}^tP_2$ of $[a, b]$ with norms less than δ, $|S(f, {}^tP_1) - S(f, {}^tP_2)| < \varepsilon$.

Theorem 5.10 (Bounded/Partition for Integrability) *173*

Let f be a bounded function defined on $[a, b]$. Then f is Riemann integrable on $[a, b]$ iff for all $\varepsilon > 0$

there exists a partition $P = \{x_i \mid 0 \le i \le n\}$ of $[a, b]$ such that

$$\sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \varepsilon$$

Theorem 5.11 (Continuous ⇒ Integrable) *175*
If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Theorem 5.12 (Monotone ⇒ Integrable) *175*
If $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Exercises

1. State the following.
   (a) Completeness Axiom
   (b) definition of convergence of a sequence
   (c) definition of Cauchy sequence
   (d) Bolzano-Weierstrass Theorem
   (e) definition of limit of a function
   (f) definition of continuity
   (g) Intermediate Value Theorem
   (h) Extreme Value Theorem
   (i) definition of differentiable
   (j) Product Rule
   (k) Quotient Rule
   (l) Chain Rule
   (m) Mean Value Theorem
   (n) definition of Riemann integrable
2. List five ways a function can be shown to be Riemann integrable.
3. Define a sequence by $x_1 = 2$ and $x_{n+1} = \dfrac{x_n}{2} + \dfrac{5}{x_n}$. Prove that the sequence converges and find its limit.
4. Show from the definition that $x_n = \dfrac{n}{n+3}$ is a Cauchy sequence and is therefore convergent.
5. Use the Squeeze Theorem to prove the sequence $\left\{\dfrac{\sin n}{n}\right\}$ converges.
6. Use the definition of limit to prove $\lim_{x \to 2}(x^2 + 3x - 4) = 6$.

7. Use the definition of limit to prove that $\lim_{x\to 0} \frac{x}{|x|} \neq -1$.

8. Use the Squeeze Theorem to prove the function $\lim_{x\to\infty} \frac{2-\cos x}{x+3}$ converges.

9. Let I be an open interval that contains the point c and suppose that f is a function that is defined on I except possibly at c. Prove that if the function f has a limit L at c then for each sequence $\{x_n\}$ in $I \setminus \{c\}$ that converges to c, the sequence $\{f(x_n)\}$ converges to L.

10. Use the definition of the derivative to find $f'(2)$ if $f(x) = x^3 + 3x^2$.

11. Find the derivative of $f(x) = \frac{\sin 2x \cos x}{x^2}$.

12. Prove that the function $f : [0, 3] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

is Riemann integrable.

Answers

1. (stated above in text)

2. (a) by definition; (b) Cauchy Criterion; (c) bounded with an oscillation (Thm. 5.10); (d) continuous; (e) monotone

3. Use the Bounded/Monotone Convergence Theorem (show, using induction, that $\{x_n\}$ is decreasing and bounded below by $\sqrt{10}$). The limit is $\sqrt{10}$.

4. Choose $\varepsilon > 0$ arbitrary. Let $N = \frac{3}{\varepsilon}$. Then for all $m, n \geq N$ with $m > n$,

$$\left|\frac{m}{m+3} - \frac{n}{n+3}\right| = \left|\frac{mn + 3m - mn - 3n}{(m+3)(n+3)}\right| = \left|\frac{3(m-n)}{(m+3)(n+3)}\right| < \frac{3m}{mn} = \frac{3}{n} < \frac{3}{N} = \varepsilon$$

Thus, the sequence is Cauchy and therefore convergent.

5. Since $-1 \leq \sin n \leq 1$ and $n \geq 1$, divide both sides by n to obtain $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. The limit as $n \to \infty$ of both $-\frac{1}{n}$ and $\frac{1}{n}$ is 0, so $\lim_{x\to\infty} \frac{\sin n}{n} = 0$.

6. Choose $\varepsilon > 0$ arbitrary. Let $\delta = \min\{\frac{\varepsilon}{8}, 1\}$. Then $|f(x) - L| = |x^2 + 3x - 10| = |x-2||x+5| < 8\delta = \varepsilon$. Thus, the function converges to 6. (Fill in the scratchwork to confirm this.)

7. Choose $\varepsilon = 1$. Then for any $\delta > 0$, if $0 < x < \delta$, $|f(x) - L| = |1 - (-1)| = 2 \neq 1$. Since we chose $x \in I$ that satisfies $0 < |x - 0| < \delta$, this contradicts the definition of the limit. Thus, the function does not converge to -1.

8. Note that $-1 \leq \cos x \leq 1 \Rightarrow 1 \geq -\cos x \geq -1 \Rightarrow 3 \geq 2 - \cos x \geq 1$. Since $x \to \infty$, assume $x + 3 > 0$. Then $\frac{3}{x+3} \geq \frac{2-\cos x}{x+3} \geq \frac{1}{x+3}$. But $\lim_{x\to\infty} \frac{3}{x+3} = \lim_{x\to\infty} \frac{1}{x+3} = 0$, so $\lim_{x\to\infty} \frac{2-\cos x}{x+3} = 0$.

9. Since $f(x) \to L$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in I$, if $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$. Also, if $\{x_n\} \to c$ then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \varepsilon$. Choose $\delta = \varepsilon$. Then $|x_n - c| < \delta$, which means $|f(x_n) - L| < \varepsilon$. Thus, $\{f(x_n)\} \to L$.

10. By definition, $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{x^3 + 3x^2 - 20}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x^2 + 5x + 10)}{x - 2} = \lim\limits_{x \to 2} (x^2 + 5x + 10) = 24$.

11. Write $f(x) = [(x^{-2}) \sin 2x] \cos x$. Then $\dfrac{d}{dx} f(x) = x^{-2} \sin 2x(-\sin x) + (\cos x)(-2x^{-3} \sin 2x + 2x^{-2} \cos 2x)$.

12. Choose $\varepsilon > 0$ arbitrary. Let $\delta < \frac{\varepsilon}{3}$. Then for any ${}^tP$ that satisfies $\|{}^tP\| < \delta$ we have $|S(f, {}^tP) - L| = |S(f, {}^tP) - 6| \leq |5\delta + 2(3 - \delta) - 6| = |5\delta + 6 - 2\delta - 6| = |3\delta| < \varepsilon$. Thus, $f(x)$ is Riemann integrable on $[0, 3]$.